The space of stability conditions for Dynkin quivers

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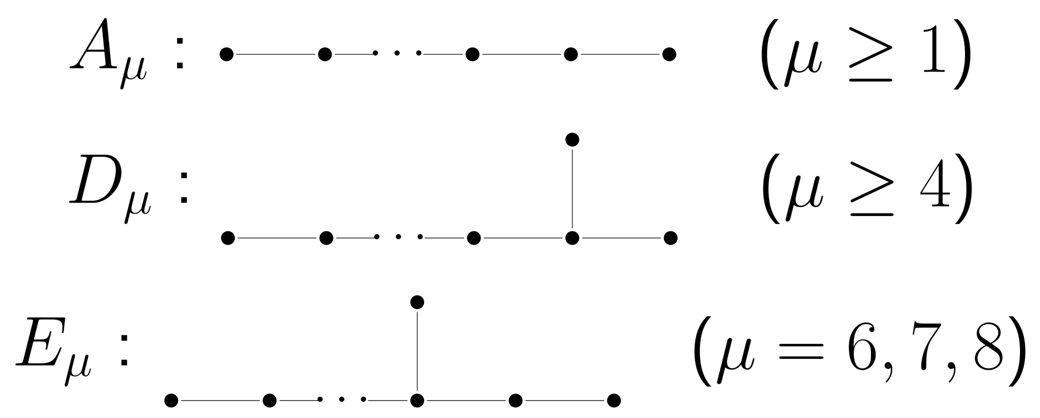
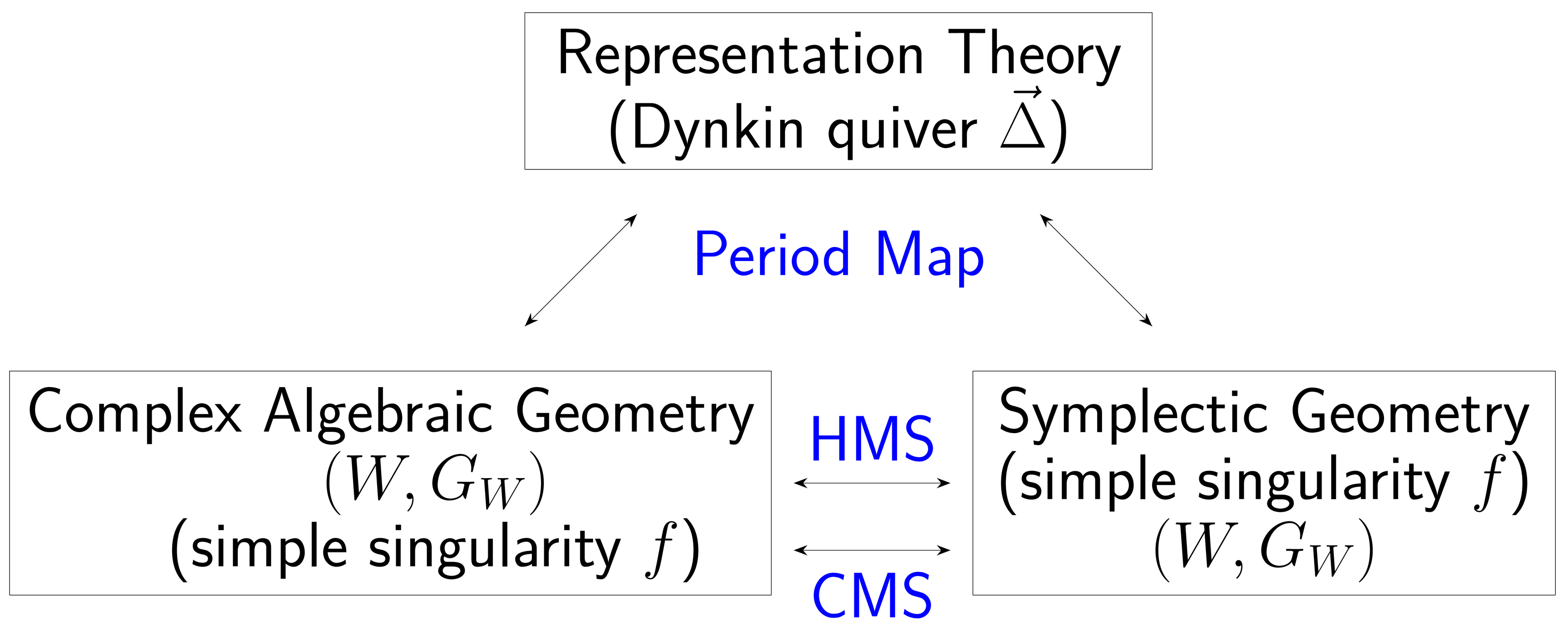
1. Mirror Symmetry

A Dynkin quiver $\vec{\Delta}$ is an oriented ADE diagram:



Mirror symmetry is an equivalence between complex algebraic geometry and symplectic geometry. Dynkin quivers and period maps play an important role in mirror symmetry. There are interesting correspondences:



Triangulated category (Homological mirror symmetry)

An additive category equipped with a triangle structure.

- $\mathcal{D}^b(\vec{\Delta})$: derived category of $\mathbb{C}\vec{\Delta}$-modules,
- $\mathcal{D}^b\mathrm{Fuk}^{\to}(f)$: derived Fukaya–Seidel category,
- $\mathrm{HMF}_S^{L_W}(W)$: category of mateix factorizations.

($\fallingdotseq \mathcal{D}^b\mathrm{Coh}(X)$: derived category of coherent sheaves on X)

$$\mathcal{D}^b(\vec{\Delta}) \simeq \mathcal{D}^b\mathrm{Fuk}^{\to}(f) \overset{MS}{\simeq} \mathrm{HMF}_S^{L_W}(W)$$

Frobenius manifold (Classical mirror symmetry)

A complex manifold equipped with a complex differential geometric structure. There are several constructions:

- $M_{(R(\vec{\Delta}),c)}$: Weyl group invariant theory,
- $M_{(f,\zeta)}$: Deformation theory with a primitive form ζ,
- $M_{(W,G_W)}$: FJRW theory (cf. Gromov–Witten theory),
- Information geometry.

$$M_{(R(\vec{\Delta}),c)} \overset{\widetilde{\Pi}_\zeta}{\cong} M_{(f,\zeta)} \overset{MS}{\cong} M_{(W,G_W)}$$

The isomorphism $\widetilde{\Pi}_\zeta$ is given by the period map.

Mirror symmetry provides rich topics in algebraic geometry:

- Semi orthogonal decompositions,
- Birational geometry,
- Gamma integral structure and Hodge theory,
- Categorical dynamics, etc.

From the viewpoint of mirror symmetry, it is expected that the Frobenius manifolds from the triangulated categories by stability conditions.

2. Stability Condition

"Definition" (Bridgeland)

A *stability condition* $(Z, \mathcal{P})$ on a triangulated category $\mathcal{D}$ consists of

- a group homomorphism $Z\colon K_0(\mathcal{D}) \longrightarrow \mathbb{C}$ and,
- a family of additive full subcategories $\mathcal{P} = \{\mathcal{P}(\phi)\}_{\phi\in\mathbb{R}}$

satisfying some axioms.

$\rightsquigarrow$ An object $E \in \mathcal{P}(\phi)$ is called σ-semistable.

Bridgeland also showed $\mathrm{Stab}(\mathcal{D})$ is a complex manifold.

Conjecture

(cf. Takahashi, Bridgeland–Qiu–Sutherland, Haiden–Katzarkov–Kontsevich)

There exists a biholomorphic map

$$\Phi\colon \mathrm{Stab}(\mathcal{D}^b(\vec{\Delta})) \xrightarrow{\cong} M_{(f,\zeta)}$$

such that the group homomorphism $Z\colon K_0(\mathcal{D}) \longrightarrow \mathbb{C}$ is given by the the exponential period map associated to ζ under the isomorphism.

In particular, $\mathrm{Stab}(\mathcal{D}^b(\vec{\Delta}))$ has a structure of a Frobenius manifold.

3. Results

Definition

A full exceptional collection $\mathcal{E} = (E_1, \ldots, E_\mu)$ in $\mathcal{D}$ satisfies

(1) $\mathrm{Hom}^\bullet_{\mathcal{D}}(E_i, E_i) \cong \mathbb{C}$ for all $i = 1, \ldots, \mu$,

(2) $\mathrm{Hom}^\bullet_{\mathcal{D}}(E_i, E_j) \cong 0$ for $i > j$ and,

(3) $\mathcal{D}$ is the smallest triangulated category containing $\mathcal{E}$.

Define the set $\mathrm{FEC}(\vec{\Delta})$, which is finite, by

$$\mathrm{FEC}(\vec{\Delta}) := \left\{ \mathcal{E} = (E_1, \ldots, E_\mu) \,\middle|\, \begin{array}{c} \mathcal{E} : \text{full exc coll in } \mathcal{D}^b(\vec{\Delta}) \\ E_i \in \mathrm{mod}(\mathbb{C}\vec{\Delta}) \end{array} \right\} \Big/ \cong .$$

It is known that the set $\mathcal{B}(f)$ of distinguished bases of vanishing cycles (up to orientation) can be identified with $\mathrm{FEC}(\vec{\Delta})$:

$$\mathcal{B}(f) \longrightarrow \mathrm{FEC}(\vec{\Delta}), \quad \mathcal{L} \mapsto \mathcal{E}_{\mathcal{L}}$$

Theorem [O]

Let $\Theta_{\mathcal{E}} := \{\sigma \in \mathrm{Stab}(\mathcal{D}^b(\vec{\Delta})) \mid E_1, \ldots, E_\mu \text{ are } \sigma\text{-stable}\}$.

The finite set $\{\Theta_{\mathcal{E}_{\mathcal{L}}}\}_{\mathcal{L}\in\mathcal{B}(f)}$ is an open covering :

$$\mathrm{Stab}(\mathcal{D}^b(\vec{\Delta})) = \bigcup_{\mathcal{L}\in\mathcal{B}(f)} \Theta_{\mathcal{E}_{\mathcal{L}}}.$$

At the "origin" of $M_{(f,\zeta)}$, we analyzed the certain stability condition constructed by Kajiura–Saito–Takahashi.

Theorem [O–Takahashi, Kajiura–Saito–Takahashi]

There is a stability condition $\sigma_0 = (Z, \mathcal{P})$ on $\mathcal{D}^b(\vec{\Delta})$ such that

(KST) Any indecomposable objects are σ_0-stable,

(OT) $Z\colon K_0(\mathcal{D}^b(\vec{\Delta})) \longrightarrow \mathbb{C}$ is given by the exponential period map associated to ζ.