The space of stability conditions for Dynkin quivers

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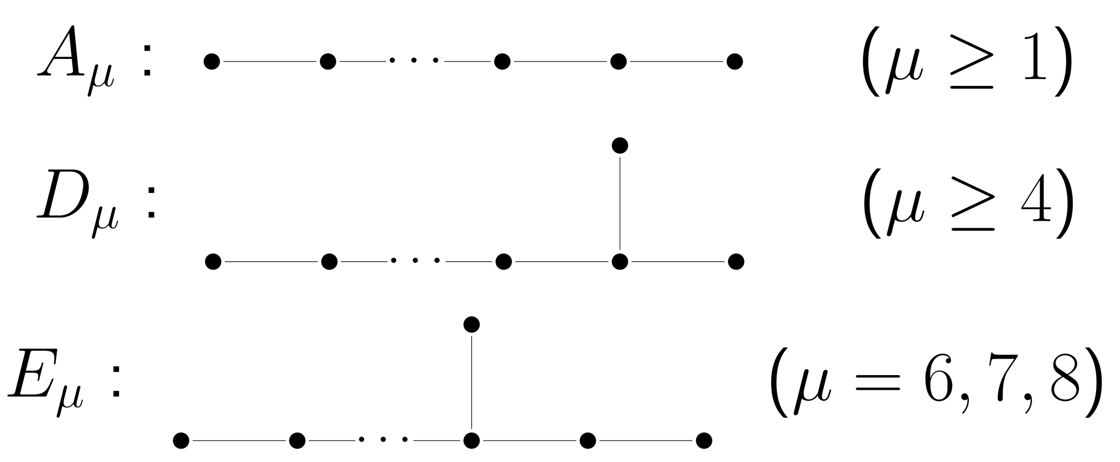
1. Dynkin Quiver

Dynkin quiver $\vec{\Delta} \overset{\text{def}}{\Longleftrightarrow}$ An oriented ADE diagram:



Dynkin quivers are closely related to polynomials of type ADE (simple singularities) $f \colon \mathbb{C}^3 \longrightarrow \mathbb{C}$.

Theorem [Seidel]

$\mathcal{D}^b(\vec{\Delta}) = \mathcal{D}^b\mathrm{mod}(\mathbb{C}\vec{\Delta})$: derived category of $\mathbb{C}\vec{\Delta}$-modules,
$\mathcal{D}^b\mathrm{Fuk}^{\rightarrow}(f)$: derived Fukaya–Seidel category for f,

$$\mathcal{D}^b(\vec{\Delta}) \simeq \mathcal{D}^b\mathrm{Fuk}^{\rightarrow}(f).$$

$\mathcal{D}^b(\vec{\Delta})$ is decomposed by a full exceptional collections.

Definition

$\mathcal{D}$: triangulated category.
A full exceptional collection $\mathcal{E} = (E_1, \ldots, E_\mu)$ in $\mathcal{D}$ satisfies
1. $\mathrm{Hom}^\bullet_{\mathcal{D}}(E_i, E_i) \cong \mathbb{C}$ for all $i = 1, \ldots, \mu$,
2. $\mathrm{Hom}^\bullet_{\mathcal{D}}(E_i, E_j) \cong 0$ for $i > j$ and,
3. $\mathcal{D}$ is the smallest triangulated category containing $\mathcal{E}$.

$$\mathrm{FEC}(\vec{\Delta}) := \left\{ \mathcal{E} = (E_1, \ldots, E_\mu) \;\middle|\; \begin{array}{c} \mathcal{E} : \text{full exc coll in } \mathcal{D}^b(\vec{\Delta}) \\ E_i \in \mathrm{mod}(\mathbb{C}\vec{\Delta}) \end{array} \right\} \Big/ \cong$$

Mutation:
The Braid group Br_μ acts on the set $\mathrm{FEC}(\vec{\Delta})$.

On the other hand,
$\mathcal{B}(f)$: the set of distinguished bases of vanishing cycles (up to orientation) for the singularity $f \colon \mathbb{C}^3 \longrightarrow \mathbb{C}$.

Picard–Lefschetz transformations:
The Braid group Br_μ also acts on the set $\mathcal{B}(f)$.

Theorem [Gusein-Zade, Seidel]

$\exists$ Br_μ-equivariant bijection

$$\mathcal{B}(f) \overset{1:1}{\longrightarrow} \mathrm{FEC}(\vec{\Delta}), \; \mathcal{L} \mapsto \mathcal{E}_{\mathcal{L}}$$

2. Stability Condition

"Definition" (Bridgeland)

A stability condition $(Z, \mathcal{P})$ on $\mathcal{D}$ consists of
- a group homomorphism $Z \colon K_0(\mathcal{D}) \longrightarrow \mathbb{C}$ and,
- a family of additive full subcategories $\mathcal{P} = \{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}}$

satisfying some axioms.

$\sigma = (Z, \mathcal{P})$: stability condition on $\mathcal{D}$.
$E \in \mathcal{D}$: σ-stable $\overset{\text{def}}{\Longleftrightarrow}$ $E \in \mathcal{P}(\phi)$ is simple for some $\phi \in \mathbb{R}$.

$$\mathrm{Stab}(\mathcal{D}) := \{\text{stability condition on } \mathcal{D}\}$$

Theorem [Bridgeland]

$\mathcal{D}$: $\mathbb{C}$-linear triangulated category of finite type
1) $\exists$ topology on $\mathrm{Stab}(\mathcal{D})$.
2) The map

$$\mathcal{Z} \colon \mathrm{Stab}(\mathcal{D}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(K_0(\mathcal{D}), \mathbb{C}), \; (Z, \mathcal{P}) \mapsto Z,$$

is local homeomorphism.
In particular, $\mathrm{Stab}(\mathcal{D})$ is a complex manifold.

Macrì studied stability conditions associated with full exceptional collections.

Based on his study, Dimitrov–Katzarkov introduced the notion of a full σ-exceptional collection for a stability condition σ.

Definition (Dimitrov–Katzarkov cf. Macrì)

$\sigma = (Z, \mathcal{P})$: stability condition on $\mathcal{D}$.
A full σ-exceptional collection $\mathcal{E} = (E_1, \ldots, E_\mu)$ satisfies
1. $E_1, \ldots, E_\mu$ are σ-stable,
2. $\mathcal{P}((0, 1])$ is generated by $\mathcal{E}$ as the extension closure.

Theorem [O] (conjectured by [Dimitrov–Katzarkov])

For any $\sigma \in \mathrm{Stab}(\mathcal{D}^b(\vec{\Delta}))$, $\exists$ full σ-exceptional collection.

As a corollary, $\mathrm{Stab}(\mathcal{D}^b(\vec{\Delta}))$ is described by $\mathcal{B}(f)$.

Theorem [O]

For $\mathcal{E} = (E_1, \ldots, E_\mu) \in \mathrm{FEC}(\vec{\Delta})$, put

$$\Theta_{\mathcal{E}} := \{\sigma \in \mathrm{Stab}(\mathcal{D}^b(\vec{\Delta})) \mid E_1, \ldots, E_\mu \text{ are } \sigma\text{-stable}\}.$$

The finite set $\{\Theta_{\mathcal{E}_{\mathcal{L}}}\}_{\mathcal{L} \in \mathcal{B}(f)}$ is an open covering :

$$\mathrm{Stab}(\mathcal{D}^b(\vec{\Delta})) = \bigcup_{\mathcal{L} \in \mathcal{B}(f)} \Theta_{\mathcal{E}_{\mathcal{L}}}.$$

3. Problem

In our setting, the homological mirror symmetry was proved by Kajiura–Saito–Takahashi and Seidel:

$\exists$ (W, G_W) : invertible polynomial with a group (Landau–Ginzburg orbifold)

s.t.

$$\mathrm{HMF}^{L_W}_S(W) \simeq \mathcal{D}^b(\vec{\Delta}) \simeq \mathcal{D}^b\mathrm{Fuk}^{\rightarrow}(f).$$

As an analogue of the Gromow–Witten theory, one can consider the Fan–Jarvis–Ruan–Witten theory for (W, G_W).

The FJRW theory for (W, G_W) is isomorphic to the Saito theory for f with a primitive form ζ as Frobenius manifolds.

Problem

How do we obtain the FJRW theory for (W, G_W) from $\mathrm{Stab}(\mathrm{HMF}^{L_W}_S(W))$?
In other words, how do we obtain the Saito theory for f with a primitive form ζ from $\mathrm{Stab}(D^b\mathrm{Fuk}^{\rightarrow}(f))$?

$\rightsquigarrow$ Need to study $\mathcal{Z} \colon \mathrm{Stab}(\mathcal{D}^b(\vec{\Delta})) \rightarrow \mathrm{Hom}_{\mathbb{Z}}(K_0(\mathcal{D}^b(\vec{\Delta})), \mathbb{C})$!